Particle creation via relaxing hypermagnetic knots

C. Adam*
Institut für theoretische Physik, Universität Karlsruhe, 76128 Karlsruhe, Germany

B. Muratori† and C. Nash‡
Department of Mathematical Physics, National University of Ireland, Maynooth, Ireland
(Received 26 July 2000; published 23 October 2000)

We demonstrate that particle production for fermions coupled chirally to an Abelian gauge field like the hypercharge field is provided by the microscopic mechanism of level crossing. For this purpose we use recent results on zero modes of Dirac operators for a class of localized hypermagnetic knots.

PACS number(s): 11.27.+d, 95.30.Cq

I. INTRODUCTION

It is well known that in chiral theories the chiral anomaly [1,2] together with some topological properties of the gauge fields open the possibility for particle-number violating processes [3]. On one hand, such processes may become relevant at high-energy experiments in the TeV range; on the other hand some version of these particle-number violating processes is believed to be the most promising candidate for explaining the dominance of matter over antimatter observed in the current universe.

These particle-number violating processes are described in a slightly different manner, depending on whether the relevant gauge fields are non-Abelian [containing an SU(2) subgroup] or Abelian. In the non-Abelian case even gauge fields interpolating between vacuum configurations at initial time $t_{\rm i}$ and final time $t_{\rm f}$ may provide particle-number violation (i.e. particle creation) provided that the pure vacuum fields at $t_{\rm i}$ and $t_{\rm f}$ have different winding numbers $n_{\rm i}$ and $n_{\rm f}$, respectively. Moreover, the total number of particles created is two times the Pontryagin index of the given gauge field in a Euclidean formulation. In Minkowski space this translates into the condition that the number of created particles is two times the number of levels (eigenvalues of the spatial but time-dependent Dirac operator) that cross zero as the gauge field evolves from the vacuum at $t_{\rm i}$ to the vacuum at $t_{\rm f}$ [4]. This introduces a microscopic description of the particle creation mechanism in the sense that each time a level crosses zero a particle is created and an anti-particle is annihilated (or vice versa, depending on the sign of the level crossing).

In the Abelian case there are no nontrivial, nonequivalent vacuum configurations, and therefore particle creation is not expected for vacuum to vacuum transitions. However, the anomaly relation still predicts particle creation, e.g., for a transition from a non-vacuum configuration (with non-zero Chern-Simons number) to a vacuum configuration [5,6]. It may for instance happen that non-trivial configurations of the Abelian hyper-charge field (hypermagnetic knots) are formed at an instance of time when there is still thermodynamic equilibrium in the universe such that any imbalance between particles and antiparticles is washed out immediately. When these hypermagnetic knots relax at later times when thermodynamic equilibrium no longer holds, particle creation may result. These features are described in detail e.g. in [7–11] and in the references quoted there.

Here the question arises whether a similar microscopic mechanism (like the level-crossing phenomenon) may be identified that is responsible for the particle creation. The problem here is that there is no simple topological feature like the index theorem [12–14] in the non-Abelian case that guarantees the existence and counts the number of zero modes. In addition, until recently there was not much information available on zero modes of the spatial (i.e. three-dimensional) Abelian Dirac operator, at all. The first example of such a zero mode was given only in 1986 in [15], and some further results including the feature of zero mode degeneracy were obtained recently in [16–21].

It is the purpose of this paper to show that in the case of the class of localized hypermagnetic knots that were discussed in [18] it is indeed the level crossing phenomenon that accounts for particle production. We shall find that the number of levels crossed exactly matches the number of particles created as predicted by the anomaly equation.

We use Minkowski space conventions, $\eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$, Greek indices are space-time indices and latin indices are space indices.

II. ANOMALY EQUATION

The Abelian anomaly equation for one fermion species is (see e.g. [22])

$$\partial^\mu(J^{\rm L}_\mu - J^{\rm R}_\mu) = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta} \tag{1}$$

where the coupling constant is absorbed into the Abelian gauge field A_μ, $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and it is assumed that the left-handed and right-handed currents couple differently. Here one could, for example, assume the hyper-charge assignments of the different particle species. However, as we are mainly interested in the general features we assume in the sequel that the coupling is purely chiral, $J^{\rm R}=0$, and we con-

*Email address: adam@maths.tcd.ie, adam@pap.univie.ac.at
†Email address: bmurator@fermi1.thphys.may.ie
‡Email address: cnash@stokes2.thphys.may.ie

sider only one particle species. Further we assume for later convenience that $A_0=0$. Then the anomaly equation may be re-written like

$$\partial^\mu J^{\rm L}_\mu = -\frac{1}{4\pi^2}\epsilon_{ijk}[\partial_0(A_i\partial_j A_k)+\partial_i(A_j\partial_0 A_k)] \tag{2}$$

(concerning the apparent sign change, we use Euclidean conventions for the purely space-like indices, i.e. $a_j b_j \equiv \vec{a}\cdot\vec{b}$). If we further assume that both currents and gauge fields are confined to a finite space region for the time interval we consider we find upon integration over all space and the time interval $[t_{\rm i},t_{\rm f}]$

$$\int d^3x(J^{\rm L}_0(t_{\rm f},\vec{x})-J^{\rm L}_0(t_{\rm i},\vec{x})) = -\frac{1}{4\pi^2}\int d^3x[(\vec{A}\cdot\vec{B})(t_{\rm f},\vec{x})-(\vec{A}\cdot\vec{B})(t_{\rm i},\vec{x})] \tag{3}$$

where the left-hand side (LHS) is the change in particle number between $t_{\rm i}$ and $t_{\rm f}$.

III. LEVEL CROSSING: SIMPLEST CASE

For the discussion of the level crossing phenomenon, let us start from the usual four-dimensional Dirac equation

$$\gamma^\mu(-i\partial_\mu - A_\mu)\psi=0. \tag{4}$$

We want to study zero energy bound states at some fixed time, therefore ψ has to be independent of time for our purposes. Further, we study a chiral theory, therefore the four-component Dirac spinor ψ has to be replaced by the two-component Weyl spinor Ψ. Assuming $A_0=0$ and choosing an appropriate representation of the gamma matrices we finally get ($\vec{\sigma}$ are the Pauli matrices)

$$\sigma_k(-i\partial_k - A_k)\Psi=0. \tag{5}$$

Here A_k consists of space components only but is still time dependent. As we want to study relaxation processes that do not change the shape of the gauge field (and hypermagnetic field) we now assume

$$A_k(t,\vec{x})=c(t)A_k(\vec{x}) \tag{6}$$

where $c(t)$ relaxes from an initial to a final value.

At this point we should describe the simplest knotted field configuration that we want to discuss in this section. The gauge field and its hypermagnetic field are ($r:=|\vec{x}|$)

$$A_l(t,\vec{x})=c(t)\frac{N_l(\vec{x})}{1+r^2} \tag{7}$$

$$B_j(t,\vec{x})=\epsilon_{jkl}\partial_k A_l(t,\vec{x})=4c(t)\frac{N_j(\vec{x})}{(1+r^2)^2} \tag{8}$$

where $\vec{N}(\vec{x})$ is the vector field

$$\vec{N}(\vec{x})=\frac{1}{(1+r^2)}\begin{pmatrix}2x_1x_3-2x_2\\ 2x_2x_3+2x_1\\ 1-x_1^2-x_2^2+x_3^2\end{pmatrix} \tag{9}$$

with unit length $\vec{N}^2=1$. In addition, there is a hyperelectric field due to the time dependence of A_k, which may however be weak if $c(t)$ changes only slowly in time.

The hypermagnetic field (8) is the simplest possible localized knotted magnetic field configuration [11]. Alternatively it may be described as the Hopf curvature of the simplest Hopf map [18,23], or it may be generated in the following way: choose the simplest nontrivial SU(2) pure gauge element $U(\vec{x})$ on compatified R^3 with winding number one (i.e., the identity map $S^3\to S^3$). Perform an Abelian projection via $G_k(\vec{x})=\mathrm{tr}\,(U^\dagger\partial_k U\sigma_3)$, then the Abelian gauge field G_k is proportional to the gauge field (7), and the magnetic field $\epsilon_{jkl}\partial_k G_l$ is proportional to the knotted hypermagnetic field (8) [24].

Now we need the following important result on zero modes of the above spatial Dirac equation (5) for the specific gauge field (7). When the factor $c(t)$ in Eq. (7) (treated now as an arbitrary constant) obeys

$$c=1+2k,\quad k=1,2,3,\ldots \tag{10}$$

then the Dirac equation (5) has precisely k (square-integrable, non-singular) zero modes. This was demonstrated by explicit construction of the zero modes in [17,18]. Further, it was proven in [20] that these are indeed all zero modes that exist for the Dirac operator (5) with gauge field (7).

For later convenience, we display the single zero mode for the simplest case, $c=3$,

$$\Psi=\frac{4}{(1+r^2)^{3/2}}(\mathbf{1}+i\vec{x}\vec{\sigma})\begin{pmatrix}1\\0\end{pmatrix}. \tag{11}$$

The construction of the higher zero modes (for $c=5,7,\ldots$) is explained in the Appendix.

Now assume that $c(t)$ is relaxed from an initial value

$$c(t_{\rm i})=1+2K \tag{12}$$

where K is assumed large, to a small (or zero) final value $c(t_{\rm f})<3$ such that $c'<0$ for $t_{\rm i}\le t\le t_{\rm f}$. Then the number L of levels crossed obviously is

$$L=\sum_{k=1}^{K}k=\frac{K(K+1)}{2}. \tag{13}$$

Finally we have to calculate the Chern-Simons number of the gauge field (7). This is easily done and leads to

$$\frac{1}{4\pi^2}\int d^3x\vec{A}(t,\vec{x})\cdot\vec{B}(t,\vec{x})=\frac{1}{4\pi^2}\int d^3x\frac{4c^2(t)}{(1+r^2)^3}=\frac{c^2(t)}{4}. \tag{14}$$

Inserting this into the anomaly equation (3) and using the value (12) for $c(t_{\rm i})$ leads to the following result for the number N of particles created:

$$N=\frac{1}{4}(c^2(t_{\rm i})-c^2(t_{\rm f}))=K(K+1)+\frac{1}{4}(1-c^2(t_{\rm f})). \quad (15)$$

Interestingly, if we want to have a precise matching $N=2L$ between the number N of created particles and the number L of level crossings then we should choose $c(t_{\rm f})=\pm 1$ instead of $c(t_{\rm f})=0$ [in fact, $c(t_{\rm f})=-1$ will turn out to be the more natural choice, see below]. However, for large K this difference becomes, of course, negligible.

IV. LEVEL CROSSING: MORE GENERAL CASES

Here we want to demonstrate that the relation $N=2L$ between the number N of particles created and the number L of levels crossed holds, in fact, for a much wider class of gauge potentials (and their hypermagnetic fields). This class of fields and their zero modes were discussed in [18], so we have to review some of these results.

In [18] the concept of Hopf maps was used, so let us briefly explain it. Hopf maps are maps $S^3\rightarrow S^2$. The third homotopy group of the two-sphere is non-trivial, $\Pi_3(S^2)=\mathbf{Z}$, therefore such maps are characterized by an integer topological index, the so-called Hopf index. Hopf maps may be expressed, e.g., by maps $\chi:\mathbf{R}^3\rightarrow\mathbf{C}$ provided that the complex function χ obeys $\lim_{|\vec{x}|\rightarrow\infty}\chi(\vec{x})=\chi_0=\text{const}$. The pre-images in $\mathbf{R}^3$ of points of the target S^2 (i.e., the pre-images of points $\chi=\text{const}$) are closed curves in $\mathbf{R}^3$ (or in the related domain S^3). Any two different closed curves are linked $N_{\rm H}$ times, where $N_{\rm H}$ is the Hopf index of the given Hopf map χ. Further, a magnetic field $\vec{\mathcal{B}}$ (the Hopf curvature) is related to the Hopf map χ via

$$\vec{\mathcal{B}}=\frac{2}{i}\frac{(\vec{\partial}\overline{\chi})\times(\vec{\partial}\chi)}{(1+\overline{\chi}\chi)^2}=2\frac{(\vec{\partial}T)\times\vec{\partial}\sigma}{(1+T)^2} \quad (16)$$

where $\chi=Se^{i\sigma}$ is expressed in terms of its modulus $S=:T^{1/2}$ and phase σ at the RHS of Eq. (16).

Mathematically, the curvature $\mathcal{F}=\frac{1}{2}\mathcal{F}_{ij}dx_i dx_j$, $\mathcal{F}_{ij}=\epsilon_{ijk}\mathcal{B}_k$, is the pullback under the Hopf map, $\mathcal{F}=\chi^*\Omega$, of the standard area two-form Ω on the target S^2. Geometrically, $\vec{\mathcal{B}}$ is tangent to the closed curves $\chi=\text{const}$ (see e.g. [23,25,26,24]). The Hopf index $N_{\rm H}$ of χ may be computed from $\vec{\mathcal{B}}$ via

$$N_{\rm H}=\frac{1}{16\pi^2}\int d^3x\vec{\mathcal{A}}\cdot\vec{\mathcal{B}} \quad (17)$$

where $\vec{\mathcal{B}}=\vec{\partial}\times\vec{\mathcal{A}}$.

The simplest (standard) Hopf map χ with Hopf index $N_{\rm H}=1$ is

$$\chi=\frac{2(x_1+ix_2)}{2x_3-i(1-r^2)} \quad (18)$$

with modulus and phase

$$T:=\overline{\chi}\chi=\frac{4(r^2-x_3^2)}{4x_3^2+(1-r^2)^2},\quad \sigma=\arctan\frac{x_2}{x_1}+\arctan\frac{1-r^2}{2x_3}. \quad (19)$$

The Hopf curvature of the simplest standard Hopf map is

$$\vec{\mathcal{B}}=\frac{16}{(1+r^2)^2}\vec{N} \quad (20)$$

i.e., it is just the magnetic field (8) with $c=4$.

Specifically, we are interested in Hopf maps that are compositions of the standard Hopf map with maps $G:S^2\rightarrow S^2$,

$$\chi_G:S^3\xrightarrow{\chi}S^2\xrightarrow{G}S^2. \quad (21)$$

Here, if G has degree (i.e. winding number) m, then the composed Hopf map χ_G has Hopf index $N_{\rm H}=m^2$.

Now the result of [18] is as follows. Construct a gauge field (and its magnetic field) by subtracting the ''background'' field $\vec{A}^{\rm B}$ from an arbitrary Hopf connection of the type (21),

$$\vec{A}^{(G)}=\vec{\mathcal{A}}^{(G)}-\vec{A}^{\rm B} \quad (22)$$

$$\vec{B}^{(G)}=\vec{\partial}\times\vec{A}^{(G)}=\vec{\mathcal{B}}^{(G)}-\vec{B}^{\rm B} \quad (23)$$

where $\vec{\mathcal{B}}^{(G)}$ is the Hopf curvature of the Hopf map (21) and

$$\vec{A}^{\rm B}=\frac{1}{1+r^2}\vec{N} \quad (24)$$

is just the simplest gauge field (7) (with $c=1$). Then the Dirac equation (5) with gauge field $\vec{A}^{(G)}$ has

$$k=2m-1 \quad (25)$$

zero modes. This implies that half-integer m corresponding to double-valued, square-root type maps $G:S^2\rightarrow S^2$ have to be allowed in order to take into account the cases when the number of zero modes is even.

Again, the zero modes for gauge fields of the type (22) have been constructed explicitly in [18], and it was proven in [20] that these are all zero modes that exist for the given gauge fields.

As in [18] we now want to consider in detail the class of maps G

$$G(z,\overline{z})=f(\overline{z}z)z^m=:g^{1/2}(\overline{z}z)e^{im\,\arg(z)} \quad (26)$$

(we use the complex variable z as a stereographic coordinate on S^2) which is indeed a map $S^2\rightarrow S^2$ with winding number m provided that $g(0)=0$ and $g(\infty)=\infty$, see [18]. Concretely, we assume

$$\lim_{|z|\rightarrow 0}g(\overline{z}z)\sim|\overline{z}z|^{c_0},\quad c_0>0 \quad (27)$$

$$\lim_{|z|\to\infty} g(\overline{z}z) \sim |\overline{z}z|^{c_\infty}, \quad c_\infty > 0. \tag{28}$$

Otherwise g is not much restricted (of course, $g \geqslant 0$ holds by definition). The corresponding Hopf map with Hopf index $N_{\rm H}=m^2$ reads

$$\chi_{g,m} = g^{1/2}(T)e^{im\sigma} \tag{29}$$

where m is integer or half-integer, and T and σ are given in Eq. (19). The Hopf curvature of the Hopf map (29) is $[' \equiv (\partial/\partial T)]$

$$\vec{\mathcal{B}}^{(g,m)} = m\frac{g'(1+T)^2}{(1+g)^2}\vec{\mathcal{B}} =: m\vec{\mathcal{B}}^{(g)} \tag{30}$$

where $\vec{\mathcal{B}}$ is the simplest Hopf curvature (20). The Hopf connection of Eq. (30) is

$$\vec{\mathcal{A}}^{(g,m)} = m\vec{\mathcal{A}}^{(g)} = m\left(\frac{4}{1+r^2}\vec{N} + \frac{1}{T}\left(\frac{1}{1+g} - \frac{1}{1+T}\right)(\vec{\partial}T)\times\vec{N}\right). \tag{31}$$

It may be checked after some algebra that indeed $\vec{\mathcal{B}}^{(g,m)} = \vec{\partial}\times\vec{\mathcal{A}}^{(g,m)}$, but we shall compute $\vec{\mathcal{A}}^{(g)}$ in a simple fashion in the Appendix. In addition, the construction of the zero modes for the gauge fields (22) with Hopf maps (29) is demonstrated explicitly in the Appendix.

Now let us construct the following time-dependent gauge fields:

$$\vec{A}^{(G)}(t) = c(t)\vec{\mathcal{A}}^{(g)} - \vec{A}^{\rm B} \tag{32}$$

$$\vec{B}^{(G)}(t) = c(t)\vec{\mathcal{B}}^{(g)} - \vec{B}^{\rm B} \tag{33}$$

where $c(t)$ relaxes from a large initial value $c(t_{\rm i})$ to zero, $c(t_{\rm f})=0$. If

$$c(t_{\rm i}) = \frac{K+1}{2} \tag{34}$$

then the total number of levels crossed between $t_{\rm i}$ and $t_{\rm f}$ is again

$$L = \sum_{k=1}^{K} k = \frac{K(K+1)}{2}. \tag{35}$$

Now we have to evaluate the Chern-Simons integral in Eq. (3),

$$\begin{aligned} N &= \frac{1}{4\pi^2}\int d^3x \vec{A}^{(G)}(t_{\rm i})\cdot\vec{B}^{(G)}(t_{\rm i}) - \frac{1}{4\pi^2}\int d^3x\vec{A}^{(G)}(t_{\rm f})\cdot\vec{B}^{(G)}(t_{\rm f}) = \frac{c^2(t_{\rm i})}{4\pi^2}\int d^3x\vec{\mathcal{A}}^{(g)}\cdot\vec{\mathcal{B}}^{(g)} - \frac{c(t_{\rm i})}{4\pi^2}\int d^3x[\vec{\mathcal{A}}^{(g)}\cdot\vec{B}^{\rm B} + \vec{A}^{\rm B}\cdot\vec{\mathcal{B}}^{(g)}] \\ &= \frac{c^2(t_{\rm i})}{4\pi^2}\int d^3x\frac{64}{(1+r^2)^3}\frac{g'(1+T)^2}{(1+g)^2} - \frac{c(t_{\rm i})}{4\pi^2}\int d^3x\left[\frac{16}{(1+r^2)^3} + \frac{16}{(1+r^2)^3}\frac{g'(1+T)^2}{(1+g)^2}\right]. \end{aligned} \tag{36}$$

Here we may use the fact that $\vec{\mathcal{B}}^{(g)}$ is a Hopf curvature with Hopf index one,

$$\frac{1}{16\pi^2}\int d^3x\vec{\mathcal{A}}^{(g)}\cdot\vec{\mathcal{B}}^{(g)} = 1 \tag{37}$$

and the fact that the two integrands in Eq. (36) that are multiplied by $c(t_{\rm i})$ are proportional to the integrand in Eq. (37) and to the simplest Chern-Simons integrand in Eq. (14), respectively.

[*Remark*: We want to remind the reader that the Chern-Simons integral is gauge invariant, although the integrand is not. Multiply valued gauge functions, which would violate gauge invariance, do not occur in simply connected spaces like Euclidean space (they have to be excluded because they lead to singular gauge potentials).]

We arrive at

$$N = 4c^2(t_{\rm i}) - c(t_{\rm i})(1+1) = 4\left(\frac{K+1}{2}\right)^2 - 2\frac{K+1}{2} = K(K+1). \tag{38}$$

Therefore, the relation $N=2L$ between the number N of particles created and the number L of levels crossed is confirmed for the class of gauge fields (32) with the knotted hypermagnetic fields (33).

Observe that only the Hopf connection part in Eq. (32) is relaxed from $c(t_{\rm i})$ to zero, whereas the ''background'' field remains unchanged. However, for sufficiently large $c(t_{\rm i})$ the difference becomes negligible. In addition, as the Hopf curvatures of all the Hopf maps (21) point into the same direction as the background field $\vec{B}^{\rm B}$ at each point in space, the hypermagnetic fields (23) [and specifically Eq. (33)] are still knotted field configurations.

V. SUMMARY

For a whole class of localized hypermagnetic knots we have demonstrated that the particle creation that is predicted by the anomaly equation is indeed provided by the microscopical mechanism of level crossing. We found a precise matching between the number of levels crossed on one hand and the number of particles created on the other hand. This precise matching leads, of course, to the immediate conjecture that this microscopic description remains true in more general cases. However, as already stated, some major results on zero modes of the Abelian Dirac operator in three dimensions were obtained only recently, therefore more mathematical investigations are necessary before this question can be finally answered.

Before ending this section, we want to briefly mention another point. For simplicity, we assumed throughout this paper that the relaxation of the hypermagnetic fields is provided by a simple factor $c(t)$ multiplying the given hypermagnetic knot configuration. However, by making use of the scale invariance of the Dirac equation (5) we may relate this relaxation to another type of relaxation that may be more realistic in cosmological applications. Scale invariance means that if a given $\Psi(x)$ and $\vec{A}(x)$ solve the Dirac equation (5), then

$$\sigma_k(-i\partial_k+\lambda A_k(\lambda x))\Psi(\lambda x)=0 \tag{39}$$

holds for arbitrary real λ. Now let us use Eq. (39) for the gauge field (32) and choose $\lambda=c^{-1}(t)$. It follows that the gauge field

$$\vec{A}^{(c,g)}(x):=\vec{\mathcal{A}}(c^{-1}(t)x)-c^{-1}(t)\vec{A}^{\mathrm{B}}(c^{-1}(t)x) \tag{40}$$

still leads to a Dirac operator with k zero modes when $c(t)=(k+1)/2$, i.e., the number of levels crossed is as in Sec IV when $c(t)$ changes from an initial value $c(t_{\mathrm{i}})$ to a final value $c(t_{\mathrm{f}})$. Further, the number of particles created remains the same, too, because the Chern-Simons integral remains the same,

$$\frac{1}{4\pi^2}\int d^3x\vec{A}^{(c,g)}(x)\cdot\vec{B}^{(c,g)}(x)=4c^2(t)-2c(t)+\frac{1}{4}. \tag{41}$$

Here

$$\vec{B}^{(c,g)}(x)=c^{-1}(t)\vec{\mathcal{B}}^{(g)}(c^{-1}(t)x)-c^{-2}(t)\vec{B}^{\mathrm{B}}(c^{-1}(t)x) \tag{42}$$

and it is especially interesting to consider a relaxation that is inverse to the one of Sec. IV, i.e., $c(t_{\mathrm{f}})>c(t_{\mathrm{i}}),\ \partial_t c(t)>0$. In this case the hypermagnetic knot (42) relaxes from a strong and rapidly varying field that is confined to a small region of space to a weak and slowly varying field that is spread over a large volume.

Further we observe that, as large $c(t)$ correspond to large numbers of levels crossed, an infra-red hypermagnetic field (i.e., a slowly varying one that is spread over a large region of space) is especially efficient in producing particles. This observation is interesting since, as is explained, e.g., in [8–11], only such infra-red hypermagnetic knots may survive sufficiently long in an electro-weak plasma before the electro-weak phase transition to be relevant for particle production in the early universe.

ACKNOWLEDGMENTS

B.M. gratefully acknowledges financial support from the Training and Mobility of Researchers scheme (TMR no. ERBFMBICT983476).

APPENDIX

In this appendix we want to construct the zero modes of the Dirac operators (5) with gauge fields (22) for the specific class of Hopf maps (29). For this purpose we shall make use of some further results of [18], so let us briefly review them. It was shown in [18] that the spinor

$$\Psi^{(M)}=e^{M/2}\Psi \tag{A1}$$

is a zero mode for some Dirac operator. Here Ψ is the simplest zero mode (11) and M is a function of T only, $M=M(T)$ [T is given in Eq. (19)]. Further it should hold that $\exp(M)>0$ for all $T<\infty$, i.e., $\exp(M)$ has no zeros at finite T. If $\exp(M)$ behaves like

$$\lim_{T\to\infty}\exp(M)\sim T^{-n_\infty} \tag{A2}$$

(where n_∞ must be a non-negative integer) then further square-integrable zero modes for the *same* Dirac operator like $\Psi^{(M)}$ are

$$\Psi_l^{(M)}=\chi^l\Psi^{(M)},\quad l=0,\ldots,n_\infty \tag{A3}$$

and there exist $k=n_\infty+1$ zero modes for this Dirac operator. The task in [18] was to calculate the Dirac operator and the related magnetic field from a given zero mode $\Psi^{(M)}$, and to show that these magnetic fields are indeed related to Hopf curvatures as indicated in Eq. (23).

Here we want to do the opposite. We assume that a Hopf curvature $\vec{\mathcal{B}}^{(g,m)}$ (i.e., a function g and a integer or half-integer number m) with Hopf index $N_{\mathrm{H}}=m^2$ is given, and we want to find M and the number of zero modes. In fact, the relation between (g,m) and (M,n_∞) was already calculated in [18] and reads

$$\frac{m}{1+g}=\frac{1}{1+t}+\frac{1}{2}TM'+\frac{1}{2}n_\infty \tag{A4}$$

$$m=1+\frac{1}{2}n_\infty \tag{A5}$$

which may be re-expressed as

$$M'=\frac{1}{T}\left[\frac{n_\infty+(2+n_\infty)T-2g}{(1+g)(1+T)}\right]-\frac{n_\infty}{T}. \tag{A6}$$

If g behaves as is assumed in Eqs. (27), (28) then Eq. (A6) is integrable at $T=0$. Further, it is integrable for finite T (even if g has a singularity or zero somewhere), which guarantees $-\infty<M<\infty$ for $T<\infty$, as is assumed above. In addition, the large T behavior is determined entirely by the second term on the RHS of Eq. (A6) and leads to $M\sim-n_\infty\ln T$ as it must be.

Now we want to use Eq. (A6) to prove expression (31) for the Hopf connection $\vec{\mathcal{A}}^{(g)}$. It was shown in [18] that $\vec{\mathcal{A}}^{(g)}$ in terms of M reads

$$\vec{\mathcal{A}}^{(g)}=\frac{4}{1+r^2}\vec{N}+\frac{1}{2}M'(\vec{\partial}T)\times\vec{N}. \tag{A7}$$

$\vec{\mathcal{A}}^{(g)}$ is a Hopf connection with Hopf index one, therefore we may insert M' from Eq. (A6) with $n_\infty=0$ into Eq. (A7), which easily reproduces Eq. (31) [a further pure gauge contribution is absent in Eq. (A7) because due to $n_\infty=0$ it holds that $\exp(M)>0 \forall T$].

Finally, we may calculate the zero modes for the fields of Sec. III (the simplest hypermagnetic knots) as an example. Here $g=T$ and

$$M=-n_\infty \ln(1+T) \tag{A8}$$

[where we chose the irrelevant integration constant such that $M(0)=0$]. Therefore, if the constant c in Eq. (10) is $c=1+2k$, the $k=n_\infty+1$ zero modes are

$$\Psi_l^{(M)}=\chi^l(1+T)^{-n_\infty/2}\Psi, \quad l=0,\dots,n_\infty \tag{A9}$$

where Ψ is given in Eq. (11).

[1] S. Adler, Phys. Rev. **177**, 2426 (1969).
[2] J. Bell and R. Jackiw, Nuovo Cimento A **60**, 47 (1969).
[3] G. 't Hooft, Phys. Rev. D **14**, 3432 (1976).
[4] N. Christ, Phys. Rev. D **21**, 1591 (1980).
[5] A. Vilenkin, Phys. Rev. D **22**, 3067 (1980).
[6] V. Rubakov and A. Tavkhelidze, Phys. Lett. **165B**, 109 (1985).
[7] M. Joyce and M. Shaposhnikov, Phys. Rev. Lett. **79**, 1193 (1997).
[8] M. Giovannini and M. Shaposhnikov, Phys. Rev. Lett. **80**, 22 (1998).
[9] M. Giovannini and M. Shaposhnikov, Phys. Rev. D **57**, 2186 (1998).
[10] M. Giovannini, Phys. Rev. D **61**, 063004 (2000).
[11] M. Giovannini, Phys. Rev. D **61**, 063502 (2000).
[12] M. F. Atiyah and I. M. Singer, Ann. Math. **87**, 596 (1968).
[13] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).
[14] R. Jackiw and C. Rebbi, Phys. Rev. D **16**, 1052 (1977).
[15] M. Loss and H.-Z. Yau, Commun. Math. Phys. **104**, 283 (1986).
[16] C. Adam, B. Muratori, and C. Nash, Phys. Rev. D **60**, 125001 (1999).
[17] C. Adam, B. Muratori, and C. Nash, Phys. Lett. B **485**, 314 (2000).
[18] C. Adam, B. Muratori, and C. Nash, Phys. Rev. D **62**, 085026 (2000).
[19] C. Adam, B. Muratori, and C. Nash, Mod. Phys. Lett. A **15**, 1577 (2000).
[20] L. Erdos and J. P. Solovej, math-ph/0001036.
[21] A. A. Balinski and W. D. Evans, math.SP/0003216.
[22] R. A. Bertlmann, *Anomalies in Quantum Field Theory* (Clarendon, Oxford, 1996).
[23] A. F. Ranada, J. Phys. A **25**, 1621 (1992).
[24] R. Jackiw and S.-Y. Pi, Phys. Rev. D **61**, 105015 (2000).
[25] L. Faddeev and A. Niemi, hep-th/9705176.
[26] R. Battye and P. Sutcliffe, Proc. R. Soc. London **A455**, 4305 (1999).